F: Flippin' Cards

Problem author: Mike de Vries

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Solution: Denote with $f(c, u)$ the expected time to orient the remaining $n - c$ cards if you have seen c cards and u of them were upside down. If $q(c, u)$ denotes the probability that the next card is upside down, then you get:

$$f(c,u) = (1 - q(c,u)) \cdot (f(c+1,u) + x) + \\ q(c,u) \cdot \min(f(c+1,u+1) + y, f(c+1,c-u) + z)$$

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Algorithm: Calculate $f(c, u)$ using dynamic programming in $\mathcal{O}(n^2)$ and we are done.

Are we done? In this algorithm, we need to know $q(c, u)$ (= the probability that the next card is upside down if u cards out of the seen c cards are upside down). How to calculate it?

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Calculation: For fixed p, the probability of u cards out of c upside down is $\binom{c}{u} \cdot p^u \cdot (1-p)^{c-u}$. Hence for random p, the probability becomes: $\int_0^1 \binom{c}{u} \cdot x^u \cdot (1-x)^{c-u} dx$

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Hence, we find:

$$q(c,u) = \frac{\int_0^1 \binom{c}{u} \cdot x^{u+1} \cdot (1-x)^{c-u} dx}{\int_0^1 \binom{c}{u} \cdot x^u \cdot (1-x)^{c-u} dx} = \frac{\int_0^1 x^{u+1} \cdot (1-x)^{c-u} dx}{\int_0^1 x^u \cdot (1-x)^{c-u} dx}$$

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How to evaluate this?

1. **Numeric:** We can apply numeric integration (for example, Simpson's Rule) to calculate this for all $q(c, u)$: doable, but can be tricky to get past time limit.

2. **Heavy Math:** Note that $\int_0^1 x^a(1-x)^b dx$ is the beta-function: it evaluates to

$$\int_0^1 x^a(1-x)^b dx = \frac{a! \cdot b!}{(a+b+1)!}$$

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Plug in and we get:

$$q(c,u) = \frac{\int_0^1 x^{u+1} \cdot (1-x)^{c-u} dx}{\int_0^1 x^u \cdot (1-x)^{c-u} dx} = \frac{(u+1)! \cdot (c-u)! \cdot (c+1)!}{u! \cdot (c-u)! \cdot (c+2)!} = \frac{u+1}{c+2}$$

3. **Beautiful Math:** Equivalent formalization: for card i, draw uniform random number x_i from $[0, 1]$, card i is upside down if $x_i < p$. Recall that p is also drawn uniformly from $[0, 1]$.

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Consequence: If we draw c cards and u are upside down, then this means that u cards x_i are put before p and $c - u$ cards are put after c. For the conditional probability $q(c, u)$ that card $c + 1$ is upside down, we must consider all ways to put card x_{c+1} in the order of $x_1, x_2, \ldots, x_c, p$ and all options are equally likely. There are $u + 1$ locations before p (in which card $c + 1$ is upside down) and $c - u + 1$ locations after p (in which card $c + 1$ is right side up), so $q(c, u) = \frac{u+1}{c+2}$.

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Fun fact: Note that this technique immediately proves the formula for the beta-function (for non-negative integers a, b).

Wrap up: We know that $q(c,u) = \frac{u+1}{c+2}$. This gives the dynamic program:

$$f(c,u) = \frac{c-u+1}{c+2} \cdot (f(c+1,u)+x) + \frac{u+1}{c+2} \cdot \min(f(c+1,u+1)+y, f(c+1,c-u)+z)$$

Calculate it from $c = n$ to $c = 0$ for all $0 \leq u \leq c$ with base case $f(n,u) = 0$ for all u. The answer is $f(0,0)$.

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Fun fact 2: For $c := \frac{z-x}{n(y-x)} < 1$, the answer is asymptotically $\left(\frac{1}{2} - \frac{1}{4}(1-c)^2\right) \cdot n + \mathcal{O}(\sqrt{n})$, which can be obtained by only flipping all cards after $\sqrt{n}$ moves if at least $\frac{c+1}{2}\sqrt{n}$ cards were upside down.

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Statistics: 0 submissions, 0 accepted